

(STRONGLY) $M - \mathcal{A}$ -INJECTIVE (FLAT) MODULES

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ABSTRACT. Let M be a left R -module and $\mathcal{A} = \{A\}_{A \in \mathcal{A}}$ be a family of some submodules of M . It is introduced the classes of (strongly) $M - \mathcal{A}$ -injective and (strongly) $M - \mathcal{A}$ -flat modules which are denoted by $(S)M - \mathcal{AI}$ and $(S)M - \mathcal{AF}$, respectively. It is obtained some characterizations of these classes and the relationships between these classes. Moreover it is investigated $(S)M - \mathcal{AI}$ and $(S)M - \mathcal{AF}$ precovers and preenvelopes of modules. It is also studied \mathcal{A} -coherent, $F\mathcal{A}$ and $P\mathcal{A}$ modules. Finally more generally we give the characterization of $S - \mathcal{AI}(F)$ modules where $\mathcal{A} = \{A\}_{A \in \mathcal{A}}$ is a family of some left R -modules.

1. INTRODUCTION

Let M be a left R -module and $\mathcal{A} = \{A\}_{A \in \mathcal{A}}$ be a family of some submodules of M where R is an associative ring with identity and $M^+ = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

In this paper the classes of $M - \mathcal{A}$ -injective and $M - \mathcal{A}$ -flat modules are studied which are denoted by $M - \mathcal{AI}$ and $M - \mathcal{AF}$ and generalizations of relative injective, \mathcal{A} -injective and \mathcal{A} -flat modules (see [2], [18], [12] and [13]). As a special case, M -min-injective and M -min-flat modules are given in [11]. Moreover we introduce the class of strongly $M - \mathcal{A}$ -injective modules which is denoted by $SM - \mathcal{AI}$ and a generalization of strongly M -injective modules (see [7]) and we introduce the class of strongly $M - \mathcal{A}$ -flat modules which is denoted by $SM - \mathcal{AF}$. Thus we have that $I(F) \subseteq SM - \mathcal{AI}(F) \subseteq M - \mathcal{AI}(F)$ where $I(F)$ is the class of all injective (flat) modules.

We prove that N is in $SM - \mathcal{AF}$ if and only if N^+ is in $SM - \mathcal{AI}$ and if M is finitely presented and \mathcal{A} -coherent, then N is in $SM - \mathcal{AI}$ if and only if N^+ is in $SM - \mathcal{AF}$. We obtain that $SM - \mathcal{AI}$ and $SM - \mathcal{AF}$ have some properties which are equivalent to the condition that M is finitely presented and \mathcal{A} -coherent. We have that $(S)M - \mathcal{AF}$ is a Kaplansky

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class and if M is finitely presented and \mathcal{A} -coherent, then $(S)M - \mathcal{A}I$ is also a Kaplansky class. Moreover if M is finitely presented and \mathcal{A} -coherent, then every left R -module has $SM - \mathcal{A}I$ preenvelope and cover and we have that $((S)M - \mathcal{A}F(I), ((S)M - \mathcal{A}F(I))^\perp)$ is a perfect cotorsion theory (where M is finitely presented, \mathcal{A} -coherent and R is in $(S)M - \mathcal{A}I$) and the cotorsion theory $({}^\perp(SM - \mathcal{A}I), SM - \mathcal{A}I)$ is complete. We also study $F\mathcal{A}$ and $P\mathcal{A}$ modules as generalizations of FS and PS modules (see [14],[22] and [10]). Finally we give some properties of the class $S - \mathcal{A}I(F)$ where $\mathcal{A} = \{A\}_{A \in \mathcal{A}}$ is a family of some left R -modules which is a generalization of the class $SM - \mathcal{A}I(F)$.

2. A GENERALIZATION OF RELATIVE INJECTIVE AND FLAT MODULES

We begin with giving the following definition:

Definition 2.1. *A left R -module T is called $M - \mathcal{A}$ -injective if for every exact sequence $0 \rightarrow A \rightarrow M \rightarrow M/A \rightarrow 0$, we have an exact sequence $\text{Hom}(M, T) \rightarrow \text{Hom}(A, T) \rightarrow 0$ for all $A \in \mathcal{A}$. $M - \mathcal{A}$ -injective modules are generalization of M -injective modules (see [2] for more detail). The class of $M - \mathcal{A}$ -injective modules is denoted by $M - \mathcal{A}I$.*

Example 2.2. *Let $M = {}_R R$ and \mathcal{A} be a family of some ideals of R . Then an $R - \mathcal{A}$ -injective module is taken as \mathcal{A} -injective (see [18]).*

Moreover let M be any left R -module and \mathcal{A} be a family of all simple submodules. Then $M - \mathcal{A}$ -injective is called $M - \min$ -injective in [11].

Definition 2.3. *A left R -module T is called a strongly $M - \mathcal{A}$ -injective if for every exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Y/X \rightarrow 0$ where $Y/X \cong M/A$ for some $A \in \mathcal{A}$, we have an exact sequence $\text{Hom}(Y, T) \rightarrow \text{Hom}(X, T) \rightarrow 0$, or equivalently $\text{Ext}_R^1(M/A, T) = 0$ for all $A \in \mathcal{A}$. The definition is generalization of strongly M -injective modules (see [7]).*

The class of strongly $M - \mathcal{A}$ -injective modules is denoted by $SM - \mathcal{A}I$.

Remark 2.4. *Let I denote the class of injective modules. Then we have that*

$$I \subseteq SM - \mathcal{A}I \subseteq M - \mathcal{A}I.$$

Let M be projective (in particular, let $M = {}_R R$). Then strongly $M - \mathcal{A}$ -injective modules and $M - \mathcal{A}$ -injective modules are identical.

Now, we explain that the inclusions can be proper with respect to M .

For example, take $M = \mathbb{Z}_6$ and $\mathcal{A} = \{<\bar{2}>, <\bar{3}>\}$. Then $M = \mathbb{Z}_6 = <\bar{2}> \oplus <\bar{3}>$ and so every \mathbb{Z} -module is $\mathbb{Z}_6 - \mathcal{A}$ -injective. But \mathbb{Z}_2 is not $S\mathbb{Z}_6 - \mathcal{A}$ -injective.

Let R be a ring and ${}_R R = I_1 \oplus I_2$ where I_1 and I_2 are left ideals of R and $\mathcal{A} = \{I_1, I_2\}$. Then every left R -module is $SR - \mathcal{A}$ -injective, but it is not necessary that every left R -module is injective.

Definition 2.5. A right R -module T is called $M - \mathcal{A}$ -flat if for every exact sequence $0 \rightarrow A \rightarrow M \rightarrow 0 \rightarrow M/A \rightarrow 0$, we have an exact sequence $0 \rightarrow T \otimes A \rightarrow T \otimes M$.

The class of $M - \mathcal{A}$ -flat modules is denoted by $M - \mathcal{A}F$.

Example 2.6. Let $M = {}_R R$ and \mathcal{A} be a family of some ideals of R . Then $R - \mathcal{A}$ -flat is taken as \mathcal{A} -flat (see [12]).

Moreover let M be any left R -module and \mathcal{A} be a family of simple submodules of M . Then $M - \mathcal{A}$ -flat is called $M - \min$ -flat in [11].

Lemma 2.7. The followings are equivalent:

- (i) Let T be a right R -module. For every exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Y/X \rightarrow 0$ where $Y/X \cong M/A$ for some $A \in \mathcal{A}$, we have an exact sequence $0 \rightarrow T \otimes X \rightarrow T \otimes Y$ is exact.
- (ii) $\text{Tor}_1^R(T, M/A) = 0$.

Proof. (ii) \Rightarrow (i) is easy. (i) \Rightarrow (ii), let $0 \rightarrow T \otimes X \rightarrow T \otimes Y \rightarrow T \otimes Y/X \rightarrow 0$ be exact. Then $(T \otimes Y)^+ \rightarrow (T \otimes X)^+ \rightarrow 0$ is exact and hence $\text{Hom}(Y, T^+) \rightarrow \text{Hom}(X, T^+) \rightarrow 0$ is exact. Thus T^+ is strongly $M - \mathcal{A}$ -injective and then $\text{Ext}_R^1(M/A, T^+) = 0$. Since $\text{Ext}_R^1(M/A, T^+) \cong \text{Hom}_{\mathbb{Z}}(\text{Tor}_1^R(T, M/A), \mathbb{Q}/\mathbb{Z})$ and \mathbb{Q}/\mathbb{Z} is an injective cogenerator, $\text{Tor}_1^R(T, M/A) = 0$. \square

Definition 2.8. A right R -module satisfying the equivalent conditions in Lemma 2.7 is called a strongly $M - \mathcal{A}$ -flat module.

The class of strongly $M - \mathcal{A}$ -flat modules is denoted by $SM - \mathcal{A}F$.

Remark 2.9. Let F denote the class of flat right R -modules. Then we have that

$$F \subseteq SM - \mathcal{A}F \subseteq M - \mathcal{A}F.$$

Let M be flat (in particular, let $M = {}_R R$) then $SM - \mathcal{A}F = M - \mathcal{A}F$.

Now, we explain that the inclusions can be proper with respect to M . Let $M = \mathbb{Z}_6$ and $\mathcal{A} = \{<\bar{2}>, <\bar{3}>\}$. Then \mathbb{Z}_2^+ is $\mathbb{Z}_6 - \mathcal{A}$ -flat which is in Lemma 2.14. But \mathbb{Z}_2^+ is not $S\mathbb{Z}_6 - \mathcal{A}$ -flat.

- Lemma 2.10.** (i) $(S)M - \mathcal{A}I$ is closed under (extensions and) direct summand and direct product.
(ii) $(S)M - \mathcal{A}F$ is closed under (extensions and) direct summand, direct sum and direct limit.

Proof. It follows from definitions. \square

Lemma 2.11. A right R -module N is (strongly) $M - \mathcal{A}$ -flat if and only if N^+ is (strongly) $M - \mathcal{A}$ -injective.

Proof. We give just the proof of that N is $M - \mathcal{A}$ -flat if and only if N^+ is $M - \mathcal{A}$ -injective. Similarly we can prove the other part.

Let $A \in \mathcal{A}$. Then $0 \rightarrow N \otimes A \rightarrow N \otimes M$ is exact if and only if $(N \otimes M)^+ \rightarrow (N \otimes A)^+ \rightarrow 0$ is exact if and only if $\text{Hom}(M, N^+) \rightarrow \text{Hom}(A, N^+) \rightarrow 0$ is exact. So N is $M - \mathcal{A}$ -flat if and only if N^+ is $M - \mathcal{A}$ -injective. \square

Definition 2.12. M is called (quotient) \mathcal{A} -coherent if for all $A \in \mathcal{A}$ (M/A) A is finitely presented.

Lemma 2.13. Let M be finitely presented and \mathcal{A} -coherent. Then M is quotient \mathcal{A} -coherent.

Proof. Let $A \in \mathcal{A}$. We have the diagram

$$\begin{array}{ccccccc}
 (\Pi R) \otimes A & \longrightarrow & (\Pi R) \otimes M & \longrightarrow & (\Pi R) \otimes M/A & \longrightarrow & 0 \\
 \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\
 \Pi A & \longrightarrow & \Pi M & \longrightarrow & \Pi M/A & \longrightarrow & 0
 \end{array}$$

By Lemma 3.2.22 in [5], α and β are isomorphism. By five lemma γ is also isomorphism. By Theorem 3.2.33 in [5], M/A is finitely presented. \square

Lemma 2.14. Let M be finitely presented left R -module and \mathcal{A} -coherent. Then the following are equivalent:

- (i) N is in $(S)M - \mathcal{A}I$.
(ii) N^+ is in $(S)M - \mathcal{A}F$.

Notice that it is not necessary to be that M is finitely presented to prove that N is in $M - \mathcal{A}I$ implies that N^+ is in $M - \mathcal{A}F$.

Proof. (i) \Leftrightarrow (ii). Since M is quotient \mathcal{A} -coherent by Lemma 2.13, by Lemma 3.60 in [16] we have the isomorphism

$$\text{Tor}_1^R(N^+, M/A) \cong \text{Ext}_R^1(M/A, N)^+.$$

So N is in $SM - \mathcal{A}I$ if and only if N^+ is in $SM - \mathcal{A}F$. Now, we will prove that N is in $M - \mathcal{A}I$ if and only if N^+ is $M - \mathcal{A}F$. We have the following commutative diagram as follow:

$$\begin{array}{ccc} N^+ \otimes A & \xrightarrow{\alpha} & N^+ \otimes M \\ \beta \downarrow & & \downarrow \theta \\ \text{Hom}(A, N)^+ & \xrightarrow{\gamma} & \text{Hom}(M, N)^+ \end{array}$$

Since A and M are finitely presented, by Lemma 3.60 in [16] β and θ are isomorphism. So N is in $M - \mathcal{A}I$ if and only if N^+ is in $M - \mathcal{A}F$. \square

The following corollaries follow from Lemma 2.11 and Lemma 2.14.

Corollary 2.15. *Let M be finitely presented and \mathcal{A} - coherent. The followings are equivalent:*

- (i) *Every $(S)M - \mathcal{A}I$ is injective.*
- (ii) *Every $(S)M - \mathcal{A}F$ is flat.*

Corollary 2.16. *Let M be finitely presented and \mathcal{A} - coherent. The followings are equivalent:*

- (i) *Every left R -module is in $(S)M - \mathcal{A}I$.*
- (ii) *Every right R -module is in $(S)M - \mathcal{A}F$.*

Example 2.17. *A simple module over a commutative ring is $SM - \mathcal{A}$ - flat if and only if it is $SM - \mathcal{A}$ - injective.*

Proof. Let $\{S_i\}_{i \in I}$ be the irredundant set of representatives of all simple R - modules and E be the injective envelope of $\bigoplus_{i \in I} S_i$. Then E is an injective cogenerator and $\text{Hom}(S, E) \cong S$ by Corollary 18.19 in [1] and Lemma 2.6 in [20], respectively. There is an isomorphism as follow

$$\text{Ext}_R^1(M/A, \text{Hom}(S, E)) \cong \text{Hom}(\text{Tor}_1^R(M/A, S), E).$$

So the requireds follow. \square

Lemma 2.18. (i) *$(S)M - \mathcal{A}F$ is closed under pure submodules and pure quotient modules.*

- (ii) *$(S)M - \mathcal{A}I$ is closed under pure submodules and pure quotient modules if M is finitely presented and \mathcal{A} - coherent.*

Proof. (i) Let N be pure submodule of an $(S)M - \mathcal{A}$ - flat module T . Then the exact sequence $0 \rightarrow N \rightarrow T \rightarrow T/N \rightarrow 0$ induces the split exact sequence $0 \rightarrow (T/N)^+ \rightarrow T^+ \rightarrow N^+ \rightarrow 0$. By Lemma 2.11, T^+ is $(S)M - \mathcal{A}$ - injective and then $(T/N)^+$ and

N^+ are also $(S)M - \mathcal{A}$ -injective. By Lemma 2.11 again, T/N and N are $(S)M - \mathcal{A}$ -flat.

(ii) The proof is similar to the part (i). □

Remark 2.19. *Let M be (finitely presented and) \mathcal{A} -coherent. Then $(S)M - \mathcal{A}I$ is closed under direct sum.*

We recall the following definitions (see [5]). A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is called perfect if every module has a \mathcal{B} -envelope and \mathcal{A} -cover. A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is complete if for any module N , there are exact sequences $0 \rightarrow N \rightarrow B \rightarrow A \rightarrow 0$ where $B \in \mathcal{B}$ and $A \in \mathcal{A}$, and $0 \rightarrow B_1 \rightarrow A_1 \rightarrow N \rightarrow 0$ where $B_1 \in \mathcal{B}$ and $A_1 \in \mathcal{A}$. Now we will give the following Theorem.

Theorem 2.20. (i) $(S)M - \mathcal{A}F$ is a Kaplansky class and also if M is finitely presented and \mathcal{A} -coherent, then $(S)M - \mathcal{A}I$ is a Kaplansky class.
(ii) Every left R -module has an $(S)M - \mathcal{A}I$ preenvelope where M is finitely presented and \mathcal{A} -coherent.
(iii) Every left R -module has an $(S)M - \mathcal{A}I$ cover where M is finitely presented and \mathcal{A} -coherent.
(iv) $((S)M - \mathcal{A}F(I), ((S)M - \mathcal{A}F(I))^\perp)$ is perfect cotorsion theory (where M is finitely presented, \mathcal{A} -coherent and R is in $(S)M - \mathcal{A}I$).
(v) The cotorsion theory $({}^\perp(SM - \mathcal{A}I), SM - \mathcal{A}I)$ is complete.

Proof. (i) It follows from Lemma 5.3.12 in [5] and Lemma 2.18.
(ii) By Proposition 6.2.1 in [5], it is understood.
(iii) Every left R -module has an $(S)M - \mathcal{A}I$ cover by Theorem 2.5 in [8] and Lemma 2.18.
(iv) $((S)M - \mathcal{A}F(I), ((S)M - \mathcal{A}F(I))^\perp)$ is a perfect cotorsion theory by Theorem 3.4 in [8].
(v) Let C be the set of representatives of all the M/A 's where $A \in \mathcal{A}$. Then $SM - \mathcal{A}I = C^\perp$. By Theorem 10 in [4], it is a complete cotorsion theory. □

Theorem 2.21. *Let M be finitely presented. Then the followings are equivalent:*

- (1) M is \mathcal{A} -coherent.
- (2) $(S)M - \mathcal{A}I$ is closed under direct limit.
- (3) $(S)M - \mathcal{A}F$ is closed under direct product (where R is a coherent ring).

- (4) A left R -module T is in $(S)M - \mathcal{AI}$ if and only if T^+ is in $(S)M - \mathcal{AF}$.
- (5) A right R -module N is in $(S)M - \mathcal{AF}$ if and only if N^{++} is in $(S)M - \mathcal{AF}$.
- (6) Every right R -module has an $(S)M - \mathcal{AF}$ preenvelope.

Proof. (1) \Rightarrow (2) Let $\{N_i\}_{i \in I}$ be any family of $(S)M - \mathcal{A}$ -injective modules. Since $(S)M - \mathcal{AI}$ is closed under pure quotient modules, by the pure exact sequence $0 \rightarrow Y \rightarrow \bigoplus N_i \rightarrow \varinjlim N_i \rightarrow 0$ by 33.9 in [21], we have that $\varinjlim N_i$ is in $(S)M - \mathcal{AI}$.

(2) \Rightarrow (1) We have the following commutative diagram

$$\begin{array}{ccc} \varinjlim \text{Hom}(M, N_i) & \xrightarrow{\alpha} & \varinjlim \text{Hom}(A, N_i) \\ \beta \downarrow & & \downarrow \theta \\ \text{Hom}(M, \varinjlim N_i) & \xrightarrow{\gamma} & \text{Hom}(A, \varinjlim N_i) \end{array}$$

where θ is monic by 24.9 in [21].

By 25.4 in [21], β is an isomorphism. Since γ is epic, θ is epic. By 25.4 in [21] again, A is finitely presented.

(1) \Rightarrow (3) Let $\{N_i\}_{i \in I}$ be a family of $M - \mathcal{A}$ -flat modules. We have the following commutative diagram

$$\begin{array}{ccc} (\prod N_i) \otimes A & \xrightarrow{\alpha} & (\prod N_i) \otimes M \\ \beta \downarrow & & \downarrow \theta \\ \prod(N_i \otimes A) & \xrightarrow{\gamma} & \prod(N_i \otimes M) \end{array}$$

By Theorem 3.2.22 in [5], β is an isomorphism. Since γ is one to one and α is one to one. Thus $\prod N_i$ is in $M - \mathcal{AI}$. (Notice that it is not necessary that M is finitely presented.)

Now, we will show that if $\{N_i\}_{i \in I}$ is a family of $SM - \mathcal{A}$ -flat modules, then $\prod N_i$ is in $SM - \mathcal{AF}$.

By Theorem 3.2.26 in [5]

$$\text{Tor}_1^R(\prod N_i, M/A) \cong \prod \text{Tor}_1^R(N_i, M/A)$$

where R is coherent and M/A is finitely presented by Lemma 2.13. So the required is found.

(3) \Rightarrow (1) Since R is flat, ΠR is in $(S)M - \mathcal{AF}$. We have the commutative diagram

$$\begin{array}{ccc} \Pi R \otimes A & \xrightarrow{\alpha} & \Pi R \otimes M \\ \beta \downarrow & & \downarrow \theta \\ \Pi A & \xrightarrow{\gamma} & \Pi M \end{array}$$

where β is epic by Lemma 3.2.21 in [5].

By Theorem 3.2.22 in [5], θ is isomorphism, so β is isomorphism. So \mathcal{A} is finitely presented by Theorem 3.22 in [5].

(1) \Rightarrow (4) follows from Lemma 2.14.

(4) \Rightarrow (5) is easy.

(5) \Rightarrow (3) Let $\{N_i\}_{i \in I}$ be a family of $(S)M - \mathcal{A}$ -flat right R -modules. Then $\oplus N_i$ is also in $(S)M - \mathcal{AF}$. By the equivalence $(\oplus N_i)^{++} \cong (\Pi N_i^+)^+$ and the part (6), $(\Pi N_i^+)^+$ is in $(S)M - \mathcal{AF}$. Since $\oplus N_i^+$ is a pure submodule of ΠN_i^+ by Lemma 1 (1) in [3], so $(\Pi N_i^+)^+ \rightarrow (\oplus N_i^+)^+ \rightarrow 0$ is split. Therefore $(\oplus N_i^+)^+$ is in $(S)M - \mathcal{AF}$. Since $\Pi N_i^{++} \cong (\oplus N_i^+)^+$, ΠN_i^{++} is in $(S)M - \mathcal{AF}$. Since ΠN_i is a pure submodule of ΠN_i^{++} by Lemma 1 (2) in [3], so ΠN_i is in $(S)M - \mathcal{AF}$.

(3) \Leftrightarrow (6) follows from Theorem 3.3 in [15] and Lemma 2.18. \square

3. $F\mathcal{A}$ AND $P\mathcal{A}$ MODULES

Definition 3.1. If A is flat (projective) for all $A \in \mathcal{A}$, then M is called an $F\mathcal{A}(P\mathcal{A})$ -module. If M is \mathcal{A} -coherent, then M is an $F\mathcal{A}$ -module if and only if M is a $P\mathcal{A}$ -module.

Proposition 3.2. The following are satisfied:

- (1) M is an $F\mathcal{A}$ -module.
- (2) Every submodule of $M - \mathcal{A}$ -flat module is again $M - \mathcal{A}$ -flat.
- (3) Every right R -module has an epic $(S)M - \mathcal{A}$ -flat preenvelope where M is \mathcal{A} -coherent (and finitely presented and R is coherent).

Then (1) \Rightarrow (2). If M is flat, then (2) \Rightarrow (1). If M is \mathcal{A} -coherent and flat, then (1) \Leftrightarrow (2) \Leftrightarrow (3).

Proof. (1) \Rightarrow (2) Let B be any $M - \mathcal{A}$ -flat module and C be a submodule of B . We have the following commutative diagram

$$\begin{array}{ccc}
C \otimes A & \xrightarrow{\alpha} & C \otimes M \\
\beta \downarrow & & \downarrow \theta \\
B \otimes A & \xrightarrow{\gamma} & B \otimes M
\end{array}$$

where β and γ are one to one. So α is one to one, and the required follows.

(2) \Rightarrow (1) Since M is flat, any $M - \mathcal{A}$ -flat is $SM - \mathcal{A}$ -flat. We have the exact sequence $0 = \text{Tor}_2^R(R, M/A) \rightarrow \text{Tor}_2^R(R/I, M/A) \rightarrow \text{Tor}_1^R(I, M/A) = 0$. So $\text{Tor}_2^{R/I}(R, M/A) = 0$. Since we have the exact sequence $0 = \text{Tor}_2^R(R/I, M/A) \rightarrow \text{Tor}_1^R(R/I, A) \rightarrow \text{Tor}_1^R(R/I, M) = 0$, so $\text{Tor}_1^R(R/I, A) = 0$, and hence A is flat.

(2) \Rightarrow (3) By Theorem 2.21, every right R -module B has an $(S)M - \mathcal{A}$ -flat preenvelope $\theta : B \rightarrow C$. By the part (2) $\theta : B \rightarrow \text{Im}\theta$ is an epic $(S)M - \mathcal{A}^F$ preenvelope of B .

(3) \Rightarrow (2) Let C be any right R -submodule of an $(S)M - \mathcal{A}$ -flat module B . Then C has an epic $(S)M - \mathcal{A}$ -flat preenvelope $\theta : C \rightarrow D$. So we have the following diagram

$$\begin{array}{ccccc}
& & B & & \\
& \nearrow i & \uparrow \gamma & & \\
C & \xrightarrow{\theta} & D & \longrightarrow & 0
\end{array}$$

where $\gamma\theta = i$. So θ is isomorphism. \square

Proposition 3.3. *The followings are satisfied.*

- (1) M is a $P\mathcal{A}$ -module.
- (2) Every quotient module of any $(S)M - \mathcal{A}$ -injective left R -module is $(S)M - \mathcal{A}$ -injective (where M is projective).
- (3) M is \mathcal{A} -coherent and every submodule of any $M - \mathcal{A}$ -flat is $M - \mathcal{A}$ -flat.
- (4) Every left R -module has a monic $(S)M - \mathcal{A}\mathcal{I}$ cover.

Then (1) \Rightarrow (2). If M is projective, then (2) \Rightarrow (1) and (4) \Rightarrow (2). If M is finitely presented, then (3) \Rightarrow (2). If A is finitely generated for all $A \in \mathcal{A}$ and M is projective, then (2) \Rightarrow (3). If M is \mathcal{A} -coherent, then (2) \Rightarrow (4).

Proof. (1) \Rightarrow (2) Let B be any $M - \mathcal{A}$ -injective left R -module and C be any submodule of B . Since $\Pi : B \rightarrow B/C$ is the canonical map and A is projective, for any $f : A \rightarrow B/C$, there exists $g : A \rightarrow B$ such that

$\Pi g = f$. So there exists $h : M \rightarrow B$ such that $hi = g$ where $i : A \rightarrow M$ is the inclusion. Then $(\Pi h)i = f$ and then (2) holds.

(2) \Rightarrow (1) Since M is projective, $SM - \mathcal{A}I = M - \mathcal{A}I$. By the exact sequence $0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0$ where N is any left R -module and E is the injective envelope of N , we have the exact sequence $0 = \text{Ext}_R^1(M/A, E/N) \rightarrow \text{Ext}_R^2(M/A, N) \rightarrow \text{Ext}_R^2(M/A, E) = 0$ and then $\text{Ext}_R^2(M/A, N) = 0$.

By the exact sequence $0 \rightarrow A \rightarrow M \rightarrow M/A \rightarrow 0$, we have that $0 = \text{Ext}_R^1(M, N) \rightarrow \text{Ext}_R^1(A, N) \rightarrow \text{Ext}_R^2(M/A, N) = 0$. So $\text{Ext}_R^1(A, N) = 0$, and then A is projective.

(3) \Rightarrow (2) Let B be any $M - \mathcal{A}$ -injective module and C be a submodule of B . Then the exact sequence $0 \rightarrow C \rightarrow B \rightarrow B/C \rightarrow 0$ induces the exact sequence $0 \rightarrow (B/C)^+ \rightarrow B^+ \rightarrow C^+ \rightarrow 0$. Since B^+ is $M - \mathcal{A}$ -flat Lemma 2.14 and so $(B/C)^+$ is $M - \mathcal{A}$ -flat. So B/C is $M - \mathcal{A}$ -injective.

(2) \Rightarrow (3) Since (2) \Rightarrow (1), by Proposition 3.2 every submodule of $M - \mathcal{A}$ -flat module is $M - \mathcal{A}$ -flat. Let N be an FP -injective left R -module and E be the injective envelope of N . Then E/N is $M - \mathcal{A}$ -injective. So we have the exact sequence $0 = \text{Ext}_R^1(M/A, E/N) \rightarrow \text{Ext}_R^2(M/A, N) \rightarrow \text{Ext}_R^2(M/A, E) = 0$ and so $\text{Ext}_R^2(M/A, N) = 0$.

By the exact sequence $0 \rightarrow A \rightarrow M \rightarrow M/A \rightarrow 0$, we have the exact sequence $0 = \text{Ext}_R^1(M, N) \rightarrow \text{Ext}_R^1(A, N) \rightarrow \text{Ext}_R^2(M/A, N) = 0$, and so $\text{Ext}_R^1(A, N) = 0$, by [6] A is finitely presented.

(2) \Rightarrow (4) Let N be a left R -module. Let $F = \sum \{x \leq N : x \in M - \mathcal{A}I\}$ and $G = \oplus \{x \leq N : x \in M - \mathcal{A}I\}$. Then there exists an exact sequence $0 \rightarrow K \rightarrow G \rightarrow F \rightarrow 0$ by [17]. Note that $G \in M - \mathcal{A}I$, so $F \in M - \mathcal{A}I$ by (2).

Let $\psi : F^1 \rightarrow N$ with $F^1 \in M - \mathcal{A}I$ be any left R -homomorphism. By (2), $\psi(F^1) \leq F$. Let $\theta : F^1 \rightarrow F$ with $\theta x = \psi x$ for $x \in F^1$. Then $i\theta = \psi$ and so $i : F \rightarrow N$ is an $M - \mathcal{A}I$ precover of N . Moreover the identity map I_F of F is the only homomorphism $g : F \rightarrow F$ such that $ig = i$, and so (4) holds.

(4) \Rightarrow (2) Let B be any $M - \mathcal{A}$ -injective module and C be a submodule of B . If E is injective envelope of C and $\phi : F \rightarrow E/C$ is the monic $M - \mathcal{A}I$ cover, then we have the following diagram

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
& & & & F & & \\
& & & \nearrow \alpha & \downarrow \phi & & \\
0 & \longrightarrow & C & \longrightarrow & E & \longrightarrow & E/C \longrightarrow 0
\end{array}$$

Then ϕ is epic, so it is isomorphism. Therefore E/C is $M - \mathcal{A}$ -injective. We have the exact sequence $0 = \text{Ext}_R^1(M/A, E/C) \rightarrow \text{Ext}_R^2(M/A, C) \rightarrow \text{Ext}_R^2(M/A, E) = 0$, and so $\text{Ext}_R^2(M/A, C) = 0$. By the exact sequence $0 = \text{Ext}_R^1(M/A, B) \rightarrow \text{Ext}_R^1(M/A, B/C) \rightarrow \text{Ext}_R^2(M/A, C) = 0$, and so $\text{Ext}_R^1(M/A, B/C) = 0$, so (2) holds. \square

Using Corollary 2.16 we can give the following proposition:

Proposition 3.4. *The following are equivalent:*

- (i) A is $M - \mathcal{A}$ -injective for all $A \in \mathcal{A}$.
- (ii) A is a direct summand of M for all $A \in \mathcal{A}$.
- (iii) Every left R -module is $M - \mathcal{A}$ -injective.
- (iv) Every right R -module is $M - \mathcal{A}$ -flat where M is finitely presented and \mathcal{A} -coherent.

Theorem 3.5. *The following are equivalent:*

- (i) Every cotorsion left R -module is in $SM - \mathcal{A}I$.
- (ii) Every pure injective left R -module is in $SM - \mathcal{A}I$.
- (iii) M/A is flat for all $A \in \mathcal{A}$.
- (iv) Every right R -module is in $SM - \mathcal{A}F$.
- (v) $\text{Tor}_1^R(R/I, M/A) = 0$ where I is any right ideal of R and for all $A \in \mathcal{A}$.
- (vi) $A \cap IM = IA$ for all right ideal I of R where M is flat.

Proof. (i) \Rightarrow (ii), (iii) \Rightarrow (i), (iii) \Leftrightarrow (iv) \Leftrightarrow (v) are trivial.

(ii) \Rightarrow (iii) Let N be any right R -module. Then N^+ is pure injective. Since $0 = \text{Ext}_R^1(M/A, N^+) \cong \text{Hom}_{\mathbb{Z}}(\text{Tor}_1^R(N, M/A), \mathbb{Q}/\mathbb{Z})$, $\text{Tor}_1^R(N, M/A) = 0$, so M/A is flat.

(v) \Leftrightarrow (vi) We have the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Tor}_1^R(R/I, M/A) & \longrightarrow & I \otimes M/A & \longrightarrow & R \otimes M/A \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
 0 & \longrightarrow & \frac{IM \cap A}{IA} & \longrightarrow & IM/IA & \longrightarrow & M/A
 \end{array}$$

Since β and γ are isomorphisms, so is α . \square

In the following proposition it is also satisfied all equivalent conditions in Theorem 3.5.

Proposition 3.6. *Let R be in $(S)M - \mathcal{AI}$. If M is in $P\mathcal{A}$ and M is flat, then M/A is flat for all $A \in \mathcal{A}$.*

Proof. Let us given that $a_j \in A$, $m_i \in M$ and $s_{ij} \in R$ such that $a_j = \sum_{i=1}^m s_{ij}m_i$ where $1 \leq j \leq n$. By the Dual Basis Lemma, there exist $\{f_k : k \in I\} \subseteq \text{Hom}(A, R)$ such that for any $c \in A$ $f_k(c) = 0$ for almost all k , and $c = \sum f_k(c)c_k$. Since R is in $(S)M - \mathcal{AI}$, there exist $g_k \in \text{Hom}(M, R)$ such that $f_k(a_j) = g_k(a_j) = g_k(\sum_{i=1}^m s_{ij}m_i) = \sum s_{ij}g_k(m_i)$. So $a_j = \sum f_k(a_j)c_k = \sum s_{ij}(\sum g_k(m_i)c_k)$. So A is pure submodule of M by Theorem 4.89 in [9]. \square

Remark 3.7. *Let \mathcal{A} be any family of some left R -modules. Then a left R -module T is called a strongly \mathcal{A} -injective(flat) if for every exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Y/X \rightarrow 0$ where $Y/X \cong S$ for some $S \in \mathcal{A}$, we have an exact sequence $\text{Hom}(Y, T) \rightarrow \text{Hom}(X, T) \rightarrow 0$ ($(T \otimes Y)^+ \rightarrow (T \otimes X)^+ \rightarrow 0$), or equivalently $\text{Ext}_R^1(S, T) = 0$ ($\text{Tor}_1^R(T, S) = 0$) for all $S \in \mathcal{A}$. The class of strongly \mathcal{A} -injective(flat) modules is denoted by $S - \mathcal{AI}(F)$ and hence $S - \mathcal{AI} = \mathcal{A}^\perp$ and $S - \mathcal{AF} = {}^\top \mathcal{A}$ (see for notations [19]). In this case $SM - \mathcal{AI}(F)$ is a special case of $S - \mathcal{AI}(F)$.*

From the proof above we see the following:

- (i) $S - \mathcal{AI}(F)$ is closed under extensions, direct summand and direct product (extensions, direct summand, direct sum and direct limit).
- (ii) A right R -module N is in $S - \mathcal{AF}$ if and only if N^+ is in $S - \mathcal{AI}$.

- (iii) *Let the modules in \mathcal{A} be finitely presented then left R - module N is in $S - \mathcal{A}I$ if and only if N^+ is in $S - \mathcal{A}F$.*
- (iv) *A simple module over commutative ring is in $S - \mathcal{A}F$ if and only if it is in $S - \mathcal{A}I$.*
- (v) *$S - \mathcal{A}F(I)$ is closed under pure submodule and pure quotient module and by Lemma 5.3.12 in [5] it is a Kaplansky class (where the modules in \mathcal{A} are finitely presented).*
- (vi) *$S - \mathcal{A}I$ is preenveloping and covering class where the modules in \mathcal{A} are finitely presented.*
- (vii) *$(S - \mathcal{A}F(I), S - \mathcal{A}F(I)^\perp)$ is a perfect cotorsion theory (where the modules in \mathcal{A} be finitely presented and R is in $S - \mathcal{A}I$).*
- (viii) *$(^\perp(S - \mathcal{A}I), S - \mathcal{A}I)$ is a complete cotorsion theory.*
- (ix) *$S - \mathcal{A}I$ is closed under direct sum and direct limit if the modules of \mathcal{A} are finitely presented.*
- (x) *$S - \mathcal{A}F$ is closed under direct product if the modules of \mathcal{A} are finitely presented and R is a coherent ring.*
- (xi) *$S - \mathcal{A}F$ is closed under direct product if and only if $S - \mathcal{A}F$ is preenveloping class.*

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